On the number of solutions of the generalized Ramanujan-Nagell equation $D_1x^2 + D_2^m = 2^{n+2}$

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Abstract Let D_1, D_2 be coprime odd integers with $\min(D_1, D_2) > 1$, and let $N(D_1, D_2)$ denote the number of positive integer solutions (x, m, n) of the equation $D_1x^2 + D_2^m = 2^{n+2}$. In this paper, we prove that $N(D_1, D_2) \le 2$ except for $N(3, 5) = N(5, 3) = 4$ and $N(13, 3) = N(31, 97) = 3$.

Keywords Exponential diophantine equation; generalized Ramanujan-Nagell equation; nu -mber of solutions; upper bound.

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§1. Introduction

Let $\mathbb{Z}, \mathbb{N}$ be the sets of all integers and positive integers respectively. Let D_1, D_2 be coprime positive odd integers with $D_2 > 1$. In 1913, S. Ramanujan [18] conjectured that all the solutions (x, n) of the equation

$$x^2 + 7 = 2^{n+2},\ x, n \in \mathbb{N}$$

are given by $(x, n) = (1, 1),\ (3, 2),\ (5, 3),\ (11, 5)$ and $(181, 13)$. Afterwards, W. Ljunggren [11] posed the same problem and T. Nagell [17] solved it in 1948. Subsequently, the equation

$$D_1x^2 + D_2 = 2^{n+2},\ x, n \in \mathbb{N} \tag{1.1}$$

is usually called the generalized Ramanujan-Nagell equation, which was solved by Y. Bugeaud and T. N. Shorey [7].

In this paper we deal with the number of solutions (x, m, n) of the equation

$$D_1x^2 + D_2^m = 2^{n+2},\ x, m, n \in \mathbb{N}, \tag{1.2}$$

which is an exponential extension of (1.1). Let $N(D_1, D_2)$ denote the number of solutions (x, m, n) of (1.2). For $D_1 = 1$, sum up the results of [4] and [9], we have:

Theorem A. If $D_1 = 1$, then $N(1, D_2) \le 1$ except for $N(1, 7) = 6$, $N(1, 23) = 2$ and $N(1, 2^r - 1) = 2$, where $r \in \mathbb{N}$ with $r > 3$.

For $D_1 > 1$, we prove a general result as follows:

Theorem B. If $D_1 > 1$, then $N(D_1, D_2) \le 2$ except for $N(3, 5) = N(5, 3) = 4$ and $N(13, 3) = N(31, 97) = 3$.

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§2. Preliminaries

Lemma 2.1.([10, Formula 1.76]) For any positive integer k and any complex numbers $\alpha,\ \beta$, we have

$$\alpha^k+\beta^k=\sum_{i=0}^{[k/2]}(-1)^i\begin{bmatrix}k\\ i\end{bmatrix}(\alpha+\beta)^{k-2i}(\alpha\beta)^i,$$

where $[k/2]$ is the integral part of $k/2$,

$$\begin{bmatrix}k\\ i\end{bmatrix}=\frac{(k-i-1)!k}{(k-2i)!i!}\in\mathbb{N},\ i=0,1,...,[k/2].$$

Lemma 2.2.([3]) Let p be an odd prime, and let X be an integer with $|X|>1$. If q is a prime divisor of $(x^p-1)/(x-1)$, then either $q=p$ or $q\equiv 1(\mathrm{mod}\ 2p)$. Further, if $p\mid (x^p-1)/(x-1)$, then $p\parallel (x^p-1)/(x-1)$.

For any nonnegative integer k, let F_k and L_k denote the k-th Fibonacci number and Lucas number respectively.

Lemma 2.3.([14, pp. 60-61])

(i) $2\mid F_kL_k$ if and only if $3\mid k$.

(ii)

$$\gcd(\mathrm{F_k},\mathrm{L_k})=\begin{cases}1, & \text{if } 3\nmid k,\\ 2, & \text{if } 3\mid k.\end{cases}$$

(iii) $L_k^2-5F_k^2=(-1)^k4$.

(iv) Every solution (u,v) of the equation

$$u^2-5v^2=\pm 4,\ u,\ v\in\mathbb{N}$$

can be expressed as $(u,v)=(L_k,F_k)$, where $k\in\mathbb{N}$.

Lemma 2.4. ([6]) The equation

$$F_k=z^n,\ k,\ z,\ n\in\mathbb{N},\ z>1,\ n>1$$

has only the solutions $(k,z,n)=(6,2,3)$ and $(12,12,2)$. The equation

$$L_k=z^n,\ k,\ z,\ n\in\mathbb{N},\ z>1,\ n>1$$

has only the solution $(k,z,n)=(3,2,2)$.

Lemma 2.5. ([16]) The equation

$$x^3+1=3y^2,\ x,\ y\in\mathbb{N}$$

has no solution (x,y).

Lemma 2.6. ([15]) Let p be an odd prime. The equation

$$x^2+x+1=3y^p,\ x,\ y,\in\mathbb{Z},\ \mid x\mid>1,\ y>1$$

has no solution (x,y).

Lemma 2.7. ([5]) The equation

$$\frac{2^r+1}{2+1}=y^n,\ r,\ y,\ n\in\mathbb{N},\ y>1,\ n>1$$

has no solution (r,y,n).

Lemma 2.8. ([13]) The equation

$$x^m-y^n=1,\ x,\ y,\ m,\ n\in\mathbb{N},\ \min(x,y,m,n)>1$$

has only the solution $(x,y,m,n)=(3,2,2,3)$.

Lemma 2.9. The equation

$$2^r+1=3^sy^n,\ r,\ s,\ y,\ n\in\mathbb{N},\ 3\nmid y,\ y>1,\ n>1 \tag{2.1}$$

has no solution (r,s,y,n).

Proof. By Lemma 2.7, (2.1) has no solution (r,s,y,n) with $s=1$.

If $s>1$ and $2\mid n$, then we have $2\nmid r$, $3\mid r$ and $(2^{r/3})^3+1=3^s(y^{n/2})^2$ by (2.1). But, since $y>1$, by Lemmas 2.5 and 2.8, it is impossible.

If $s>1$ and $2\nmid n$, then $2\nmid r$, $3\mid r$ and n has an odd prime divisor p. Since $2^r+1=(2^{r/3}+1)(2^{2r/3}-2^{r/3}+1)$ and $\gcd(2^{r/3}+1,\ 2^{2r/3}-2^{r/3}+1)=3$, we get from (2.1) that $2^{r/3}+1=3^{s-1}a^p$ and

$$2^{2r/3}-2^{r/3}+1=3b^p \tag{2.2}$$

where $a,b\in\mathbb{N}$ with $ab=y^{n/p}$. But, since $2^{r/3}>1$, by Lemma 2.6, (2.2) is impossible. Thus, the lemma is proved.

Lemma 2.10. The equation

$$2^r-1=3^sy^n,\ r,\ s,\ y,\ n\in\mathbb{N},\ 3\nmid y,\ y>1,\ n>1 \tag{2.3}$$

has no solution (r,s,y,n).

Proof. We see from (2.3) that r must be even. Since $\gcd(2^{r/2}+1,2^{r/2}-1)=1$, by (2.3), we have

$$2^{r/2}+1=\begin{cases}3^sa^n,\\ b^n,\end{cases}\quad 2^{r/2}-1=\begin{cases}b^n,\\ 3^sa^n,\end{cases}\quad y=ab,\ a,b\in\mathbb{N}, \tag{2.4}$$

However, since $y>1$ and $3\nmid y$, by Lemma 2.8, (2.4) is impossible. Thus, the lemma is proved.

Lemma 2.11. The equation

$$2^r\cdot 3^s+1=y^n,\ r,\ s,\ y,\ n\in\mathbb{N},\ y>1,\ n>1 \tag{2.5}$$

has only the solutions $(r,s,y,n)=(3,1,5,2),\ (4,1,7,2)$ and $(5,2,17,2)$.

Proof. If $2\mid n$, since $2\nmid y$, then we have $r\geq 3$ and $\gcd(y^{n/2}+1,y^{n/2}-1)=2$. Hence, by (2.5), we get

$$y^{n/2}+1=\begin{cases}2^{r-1},\\ 2\cdot 3^s,\end{cases}\quad y^{n/2}-1=\begin{cases}2\cdot 3^s,\\ 2^{r-1},\end{cases} \tag{2.6}$$

whence we obtain

$$1=\begin{cases}2^{r-2}-3^s,\\ 3^s-2^{r-2}.\end{cases} \tag{2.7}$$

Apply Lemma 2.8 to (2.7), we get $(r,s,y,n)=(3,1,5,2),\ (4,1,7,2)$ and $(5,2,17,2)$ by (2.6).

If $2\nmid n$, since $n>1$, then n has an odd prime divisor p. By (2.5), we get

$$2^r\cdot 3^s=y^n-1=(z-1)(z^{p-1}+z^{p-2}+...+1),\ z=y^{n/p}. \tag{2.8}$$

Since $2\nmid z^{p-1}+z^{p-2}+...+1$ and $3\not\equiv 1(\bmod\ 2p)$, by Lemma 2.2, we see from (2.8) that $3\geq z^{p-1}+z^{p-2}+...+1$, a contradiction. Thus, the lemma is proved.

Lemma 2.12. The equation

$$2^r\cdot 3^s-1=y^n,\ r,\ s,\ y,\ n\in\mathbb{N},\ y>1,\ n>1 \tag{2.9}$$

has no solution (r,s,y,n).

Proof. Since $(-1/3)=-1$, where $(*/*)$ is the Jacobi symbol, we see from (2.9) that n must be odd. Since $n>1$, n has an odd prime divisor p, and by (2.9), we have

$$2^r\cdot 3^s=y^n+1=(z+1)(z^{p-1}-z^{p-2}+...+1),\ z=y^{n/p}. \tag{2.10}$$

Apply Lemma 2.2 to (2.10), we get $3\geq z^{p-1}-z^{p-2}+...+1$, a contradiction. Thus, the lemma is proved.

Lemma 2.13. ([7, Lemma 1]) If the equation

$$D_1X^2+D_2Y^2=2^{Z+2},\ X,\ Y,\ Z\in\mathbb{Z},\ \gcd(X,Y)=1,\ Z>0 \tag{2.11}$$

has solutions (X,Y,Z), then it has a unique positive integer solution (X_1,Y_1,Z_1) satisfying $Z_1\leq Z$, where Z through all solutions (X,Y,Z) of (2.11). Such (X_1,Y_1,Z_1) is called the least solution of (2.11). Every solution (X,Y,Z) of (2.11) can be expressed as

$$Z=Z_1t,\quad t\in\mathbb{N},\ 2\nmid t \text{ if } D_1>1,$$

$$\frac{X\sqrt{D_1}+Y\sqrt{-D_2}}{2}=\lambda_1\left(\frac{X_1\sqrt{D_1}+\lambda_2Y_1\sqrt{-D_2}}{2}\right)^t,\ \lambda_1,\ \lambda_2\in\{\pm1\}.$$

By Lemma 2.13, we can obtain the following lemma immediately.

Lemma 2.14. If (X,Y,Z) and (X',Y',Z') are two positive integer solutions of (2.11) with $(X,Y,Z)\neq(X',Y',Z')$, then $Z\neq Z'$.

Lemma 2.15.([8, Lemma 3]) Let $\min(D_1,D_2)>1$ and $D=D_1D_2$. If (2.11) has solutions (X,Y,Z), then the equation

$$X'^2+DY'^2=2^{Z'+2},\ X',\ Y',\ Z'\in\mathbb{Z},\ \gcd(X',Y')=1,\ Z'>0 \tag{2.12}$$

has solutions (X',Y',Z'). Moreover, its least solution (X_1',Y_1',Z_1') satisfies $X_1'=\frac{1}{2}\mid D_1X_1^2-D_2Y_1^2\mid$, $Y_1'=X_1Y_1$ and $Z_1'=2Z_1$, where (X_1,Y_1,Z_1) is the least solution of (2.11).

Lemma 2.16. Let $\min(D_1,D_2)>1$ and $D=D_1D_2$. If (2.1) has a solution (X,Y,Z), then (2.2) has no solution (X',Y',Z') with $Z'=Z$.

Proof. By Lemma 2.13, if (X,Y,Z) and (X',Y',Z') are solutions of (2.11) and (2.12), then we have

$$Z=Z_1t,\ t\in\mathbb{N},\ 2\nmid t \tag{2.13}$$

and

$$Z'=Z_1't',\ t'\in\mathbb{N}, \tag{2.14}$$

where (X_1,Y_1,Z_1) and (X_1',Y_1',Z_1') are least solutions of (2.11) and (2.12) respectively. Further, by Lemma 2.15, we have $Z_1'=2Z_1$. Substituting it into (2.14), we get $Z'=2Z_1t'$. Since $2\nmid t$, we obtain $Z'\neq Z$ by (2.13). Thus, the lemma is proved.

Lemma 2.17. Let D be a positive integer. Further, let (2.12) have solutions (X',Y',Z') and (X_1',Y_1',Z_1') is its least solution. If (y,z) is a solution of the equation

$$1+Dy^2=2^{z+2},\ y,\ z\in\mathbb{N}, \tag{2.15}$$

then $X_1'=1$ and $(y,z)=(Y_1',Z_1')$ except for $D=7$ and $(y,z)=(3,4)$.

Proof. Under the assumption, (2.12) has the solution $(X',Y',Z')=(1,y,z)$. By Lemma 2.13, we have

$$z=Z_1't,\ t\in\mathbb{N}, \tag{2.16}$$

$$\frac{1+y\sqrt{-D}}{2}=\lambda_1\left(\frac{X_1'+\lambda_2Y_1'\sqrt{-D}}{2}\right)^t,\ \lambda_1,\ \lambda_2\in\{\pm1\}. \tag{2.17}$$

If $2\mid t$, let

$$\frac{a+b\sqrt{-D}}{2}=\left(\frac{X_1'+\lambda_2Y_1'\sqrt{-D}}{2}\right)^{t/2}. \tag{2.18}$$

By Lemma 2.13, then $a,\ b$ are integers satisfying

$$a^2+Db^2=2^{Z_1't/2+2}=2^{z/2+2},\ \gcd(a,b)=1. \tag{2.19}$$

Substituting (2.18) into (2.17), we get

$$a^2-Db^2=2\lambda_1,\ y=ab\lambda_1,\ \lambda_1\in\{\pm1\}. \tag{2.20}$$

The combination of (2.19) and the first equality of (2.20) yields $\lambda_1=1$ and $a^2=2^{z/2+1}+1$. Hence, by Lemma 2.8, we get $a=3,\ z=4,\ D=7,\ b=1$ and $y=3$ by (2.20).

If $2\nmid t$ and $t>1$, let

$$\alpha=\frac{X_1'+Y_1'\sqrt{-D}}{2},\ \beta=\frac{X_1'-Y_1'\sqrt{-D}}{2}, \tag{2.21}$$

then from (2.17) we get

$$1=\lambda_1(\alpha^t+\beta^t). \tag{2.22}$$

Since $\alpha+\beta=X_1'$ and $\alpha\beta=2^{Z_1'}$ by (2.21), apply Lemma 2.1 to (2.22) we have

$$1=\lambda_1\sum_{i=0}^{(t-1)/2}(-1)^i\begin{bmatrix}t\\ i\end{bmatrix}(\alpha+\beta)^{t-2i}(\alpha\beta)^i$$

$$= \lambda_1 X_1' \sum_{i=0}^{(t-1)/2} (-1)^i \begin{bmatrix} t \\ i \end{bmatrix} X_1'^{t-2i-1} 2^{Z_1' i}. \tag{2.23}$$

From (2.23), we obtain $X_1' = 1$, $\lambda_1 = 1$ and

$$t = \begin{bmatrix} t \\ 1 \end{bmatrix} = 2^{Z_1'} \sum_{j=2}^{(t-1)/2} (-1)^j \begin{bmatrix} t \\ j \end{bmatrix} 2^{Z_1'(j-2)},$$

a contradiction. It implies that $t = 1$ if $2 \nmid t$. Thus, by (2.16) and (2.17), we get $X_1' = 1$ and $(y, z) = (Y_1', Z_1')$. The lemma is proved.

Let $\alpha,\ \beta$ be algebraic integers. If $(\alpha+\beta)^2$ and $\alpha\beta$ are nonzero coprime integer and α/β is not a root of unity, then $(\alpha,\ \beta)$ is called a Lehmer pair. Let $a = (\alpha+\beta)^2$ and $b = \alpha\beta$. Then we have

$$\alpha = \frac{1}{2}(\sqrt{a} + \lambda\sqrt{c}),\ \beta = \frac{1}{2}(\sqrt{a} - \lambda\sqrt{c}),\ \lambda \in \{-1, 1\},$$

where $c = a^2 - 4b$. The pair (a, c) is called the parameter of the Lehmer pair (α, β). Two Lehmer pairs (α_1, β_1) and (α_2, β_2) are equivalent if $\alpha_1/\alpha_2 = \beta_1/\beta_2 \in \{\pm 1, \pm\sqrt{-1}\}$. Given a Lehmer pair (α, β), one defines the corresponding Lehmer numbers by

$$L_k(\alpha, \beta) = \begin{cases} \dfrac{\alpha^k - \beta^k}{\alpha - \beta}, & 2 \nmid k, \\ \dfrac{\alpha^k - \beta^k}{\alpha^2 - \beta^2}, & 2 \mid k, \end{cases} \quad k \in \mathbb{N}. \tag{2.24}$$

Lehmer numbers are nonzero integers. For equivalent Lehmer pairs (α_1, β_1) and (α_2, β_2), we have $L_k(\alpha_1, \beta_1) = \pm L_k(\alpha_2, \beta_2)$ $(k \in \mathbb{N})$. A prime q is called a primitive divisor of $L_k(\alpha, \beta)$ $(k > 1)$ if $p \mid L_k(\alpha, \beta)$ and $p \nmid (\alpha^2 - \beta^2)^2 L_1(\alpha, \beta) ... L_{k-1}(\alpha, \beta)$. A Lehmer pair (α, β) such that $L_k(\alpha, \beta)$ has no primitive divisor will be called a k-defective Lehmer pair. Further, a positive integer k is called totally non-defective if no Lehmer pair is k-defective.

Lemma 2.18. ([1],[9]) Let k satisfy $6 < k \le 30$ and $2 \nmid k$. Then, up to equivalence, all parameters of k-defective Lehmer pairs are given as follows:

(i) $k = 7$, $(a, c) = (1, -7),\ (1, -19),\ (3, -5),\ (5, -7),\ (13, -3),\ (14, -22)$.

(ii) $k = 9$, $(a, c) = (5, -3),\ (7, -1),\ (7, -5)$.

(iii) $k = 13$, $(a, c) = (1, -7)$.

(iv) $k = 15$, $(a, c) = (7, -1),\ (10, -2)$.

Lemma 2.19. ([2], Theorem 1.4) If $k > 30$, then k is totally non-defective.

§3. Further lemmas on the solutions of (1.2)

Let $D_1 > 1$. We first consider the solutions (x, m, n) of (1.2) with $2 \nmid m$. Then (2.11) has the solution

$$(X, Y, Z) = (x,\ D_2^{(m-1)/2}, n). \tag{3.1}$$

Since $\min(D_1, D_2) > 1$, apply Lemma 2.13 to (3.1), we get

$$n = Z_1 t,\ t \in \mathbb{N},\ 2 \nmid t, \tag{3.2}$$

$$\frac{x\sqrt{D_1}+D_2^{(m-1)/2}\sqrt{-D_2}}{2}=\lambda_1\left(\frac{X_1\sqrt{D_1}+\lambda_2 Y_1\sqrt{-D_2}}{2}\right)^t,\ \lambda_1,\ \lambda_2\in\{\pm 1\}. \tag{3.3}$$

where (X_1,Y_1,Z_1) is the least solution of (2.11). Let

$$\alpha=\frac{X_1\sqrt{D_1}+Y_1\sqrt{-D_2}}{2},\ \beta=\frac{X_1\sqrt{D_1}-Y_1\sqrt{-D_2}}{2}. \tag{3.4}$$

Since $X_1,\ Y_1$ and Z_1 are positive integers satisfying

$$D_1X_1^2+D_2Y_1^2=2^{Z_1+2},\ \gcd(D_1X_1^2,D_2Y_1^2)=1, \tag{3.5}$$

α and β are roots of $z^4-\frac{1}{2}(D_1X_1^2-D_2Y_1^2)z^2+2^{2Z_1}=0$. Notice that $(\alpha+\beta)^2=D_1X_1^2$ and $\alpha\beta=2^{Z_1}$ are coprime positive integers, and $\alpha/\beta=(\frac{1}{2}(D_1X_1^2-D_2Y_1^2)+X_1Y_1\sqrt{-D_1D_2})/2^{Z_1}$is not a root of unity. Then (α,β) is a Lehmer pair with parameter $(D_1X_1^2,-D_2Y_1^2)$. Let $L_k(\alpha,\beta)$ $(k\in\mathbb{N})$ denote the corresponding Lehmer numbers defined as in (2.24). By (3.3) and (3.4), we have

$$D_2^{(m-1)/2}=Y_1\mid L_t(\alpha,\beta)\mid. \tag{3.6}$$

Since $(\alpha^2-\beta^2)^2=-D_1D_2X_1^2Y_1^2$, we find from (3.6) that the Lehmer number $L_t(\alpha,\beta)$ has no primitive divisor. Therefore, we have the following result.

Lemma 3.1. If $D_1>1$, then all solutions (x,m,n) of (1.2) with $2\nmid m$ are given as follows:

(i) $t=9$, $(D_1,D_2)=(5,3),\ (x,m,n)=(19,5,9)$.

(ii) $t=7$, $(D_1,D_2)=(3,5),\ (x,m,n)=(13,1,7)$.

(iii) $t=7$, $(D_1,D_2)=(13,3),\ (x,m,n)=(71,1,14)$.

(iv) $t=5$, $(D_1,D_2)=(5,3),\ (x,m,n)=(5,1,5)$.

(v) $t=5$, $(D_1,D_2)=(21,11),\ (x,m,n)=(79,1,15)$.

(vi) $t=5$, $(D_1,D_2)=(3,29),\ (x,m,n)=(209,1,15)$.

(vii) $t=5$, $(D_1,D_2)=(3,5),\ (x,m,n)=(1,3,5)$.

(viii) $t=5$, $(D_1,D_2)=(11,5),\ (x,m,n)=(19,3,10)$.

(ix) $t=3$,

$$D_1X_1^2=2^{Z_1}-\lambda,\ D_2=3\cdot 2^{Z_1}+\lambda,\ \lambda\in\{\pm 1\}, \tag{3.7}$$

$(x,m,n)=(X_1(2^{Z_1+1}-\lambda),1,3Z_1)$.

(x) $t=3$, $(D_1,D_2)=(5,3),\ (x,m,n)=(1,3,3)$.

(xi) $t=3$, $(D_1,D_2)=(13,3),\ (x,m,n)=(1,5,6)$.

(xii) $t=1$, $Y_1=D_2^{(m-1)/2},\ (x,m,n)=(X_1,m,Z_1)$.

Proof. By Lemma 2.19, we have $t\le 30$. Further, since $2\nmid t$, by Lemma 2.18, if $7\le t\le 30$, then (1.2) has only the solutions (i), (ii) and (iii) satisfying $2\nmid m$.

For $t=5$, apply Lemma 2.1 to (3.6), we get

$$D_2^{(m-1)/2}=Y_1\mid (D_2Y_1^2)^2-5\cdot 2^{Z_1}(D_1Y_1^2)+5\cdot 2^{2Z_1}\mid. \tag{3.8}$$

If $m=1$, then from (3.8) we get $Y_1=1$ and

$$D_2^2-5\cdot 2^{Z_1}D_2+5\cdot 2^{2Z_1}=(D_2-5\cdot 2^{Z_1-1})^2-5\cdot 2^{2Z_1-2}=\lambda,\ \lambda\in\{\pm 1\}. \tag{3.9}$$

When $Z_1 = 1$, by (3.9), we have $| D_2 - 5 \cdot 2^{Z_1 - 1} | = | D_2 - 5 | = 2$. It implies that $D_2 = 3$ or 7, and by (3.5), $X_1 = 1$ and $D_1 = 5$ or 1. Since $D_1 > 1$, we get $(D_1, D_2) = (5, 3)$ and the solution (iv).

When $Z_1 > 1$, by (3.9), we get

$$(D_2 - 5 \cdot 2^{Z_1 - 1})^2 - 5(2^{Z_1 - 1})^2 = 1. \tag{3.10}$$

Apply Lemma 2.3 to (3.10), we have

$$| D_2 - 5 \cdot 2^{Z_1 - 1} | = \frac{1}{2} L_{6I+6},\ 2^{Z_1 - 1} = \frac{1}{2} F_{6I+6},\ I \in \mathbb{Z},\ I \geq 0. \tag{3.11}$$

Further, by Lemma 2.4, we see from the second equality of (3.11) that $I = 0$ and $Z_1 = 3$. Hence, by the first equality of (3.11), we get $D_2 = 11$ or 29. Further, by (3.5), we have $X_1 = 1$ and $D_1 = 21$ or 3. Thus, by (3.3), the solutions (v) and (vi) are obtained.

If $m > 1$ and $5 \nmid D_2$, since $\gcd(D_2, 5 \cdot 2^{Z_1}) = 1$, then from (3.8) we get $Y_1 = D_2^{(m-1)/2}$ and

$$(D_2^m - 5 \cdot 2^{Z_1 - 1})^2 - 5(2^{Z_1 - 1})^2 = \lambda,\ \lambda \in \{\pm 1\}. \tag{3.12}$$

Since $m > 1$ and $\min(D_1, D_2) > 1$, using the same method as in the case $m = 1$, we can prove that (3.12) is impossible.

If $m > 1$ and $5 \mid D_2$, then we have

$$Y_1 = \frac{1}{5} D_2^{(m-1)/2} \tag{3.13}$$

and

$$(2^{Z_1 - 1})^2 - 5\left(\frac{1}{125} D_2^m - 2^{Z_1 - 1}\right)^2 = \lambda,\ \lambda \in \{\pm 1\}. \tag{3.14}$$

When $Z_1 = 1$, by (3.14), we get $\frac{D_2^m}{125} - 1 = 0$, and hence, we have $D_2 = 5$ and $m = 3$. It implies that $(D_1, D_2) = (3, 5)$ and the solution (vii) is obtained.

When $Z_1 = 2$, by (3.14), we have $|\frac{1}{125} D_2^m - 2| = 1$, whence we get $D_2 = 5$ and $m = 3$. Further, by (3.3), (3.5) and (3.13), we obtain $X_1 = Y_1 = 1$, $D_1 = 11$ and the solution (viii).

For $t = 3$, by (3.8), we have

$$D_2^{(m-1)/2} = Y_1 |D_2 Y_1^2 - 3 \cdot 2^{Z_1}|. \tag{3.15}$$

If $m = 1$, then from (3.15) we get $Y_1 = 1$ and $D_2 = 3 \cdot 2^{Z_1} + \lambda$, where $\lambda \in \{\pm 1\}$. Hence, by (3.5), D_1, D_2 satisfy (3.7) and the solution (ix) is obtained.

If $m > 1$ and $3 \nmid D_2$, then we have $Y_1 = D_2^{(m-1)/2}$ and

$$D_2^m = 3 \cdot 2^{Z_1} + \lambda,\ \lambda \in \{\pm 1\}. \tag{3.16}$$

However, since $D_2 > 1, m > 1$ and $2 \nmid m$, by Lemmas 2.11 and 2.12, (3.18) is impossible.

If $m > 1$ and $3 \mid D_2$, then we have

$$Y_1 = \frac{1}{3} D_2^{(m-1)/2} \tag{3.17}$$

and

$$\frac{1}{27} D_2^m - 2^{Z_1} = \lambda,\ \lambda \in \{\pm 1\}. \tag{3.18}$$

Let $D_2=3^I D$, where $I,D\in\mathbb{N}$ with $3\nmid D$. Then (3.18) can be written as

$$3^{Im-3}D^m-2^{Z_1}=\lambda,\ \lambda\in\{\pm1\}. \tag{3.19}$$

When $D=1$, by (3.19), we have

$$3^{Im-3}-2^{Z_1}=\lambda,\ \lambda\in\{\pm1\}. \tag{3.20}$$

Since $m>1$ and $2\nmid m$, apply Lemma 2.8 to (3.20), we get $(I,m,Z_1,\lambda)=(1,3,1,-1)$ and $(1,5,3,1)$. Therefore, by (3.3), (3.5) and (3.17), the solutions (x) and (xi) are obtained.

When $D>1$, by Lemmas 2.9 and 2.10, (3.19) is impossible.

For $t=1$, by (3.2) and (3.3), the solutions (xii) is obtained. To sum up, the lemma is proved.

Let $N_1(D_1,D_2)$ denote the number of solutions (x,m,n) of (1.2) with $2\nmid m$. By Lemma 3.1, we can obtain the following lemma immediately.

Lemma 3.2. If $D_1>1$, then $N_1(D_1,D_2)\le 1$ except for the following cases:

(i) $N_1(3,5)=4,(x,m,n)=(1,1,1),(3,1,3),(1,3,5)$ and $(13,1,7)$.

(ii) $N_1(5,3)=3,(x,m,n)=(1,1,1),(1,3,3),(5,1,5)$ and $(19,5,9)$.

(iii) $N_1(13,3)=3,(x,m,n)=(1,1,2),(1,5,6)$ and $(71,1,14)$.

(iv) $N_1(11,5)=2,(x,m,n)=(1,1,2)$ and $(19,3,10)$.

(v) $N_1(21,11)=2,(x,m,n)=(1,1,3)$ and $(79,1,15)$.

(vi) $N_1(3,29)=2,(x,m,n)=(1,1,3)$ and $(209,1,15)$.

(v) If D_1 and D_2 satisfy (3.7) with $Z_1\ge 2$, then $N_1(D_1,D_2)=2$, $(x,m,n)=(X_1,1,Z_1)$ and $(X_1(2^{Z_1+1}-\lambda),1,3Z_1)$.

We next consider the solutions (x,m,n) of (1.2) with $2|m$. Then the equation

$$D_1{X'}^2+D_2^2{Y'}^2=2^{Z'+2},X',Y',Z'\in\mathbb{Z},\gcd(X',Y')=1,Z'>0 \tag{3.21}$$

has the solution

$$(X',Y',Z')=(x,D_2^{(m-2)/2},n). \tag{3.22}$$

Since $\min(D_1,D_2)>1$, apply Lemma 2.13 to (3.22), we have

$$n=Z_1't',t'\in\mathbb{N},2\nmid t', \tag{3.23}$$

$$\frac{x\sqrt{D_1}+D_2^{(m-2)/2}\sqrt{-D_2^2}}{2}=\lambda_1\left(\frac{X_1'\sqrt{D_1}+\lambda_2 Y_1'\sqrt{-D_2^2}}{2}\right)^{t'},\ \lambda_1,\lambda_2\in\{\pm1\}, \tag{3.24}$$

where (X_1',Y_1',Z_1') is the least solution of (3.21). Let

$$\alpha'=\frac{X_1'\sqrt{D_1}+Y_1'\sqrt{-D_2^2}}{2},\ \beta'=\frac{X_1\sqrt{D_1}-Y_1'\sqrt{-D_2^2}}{2}. \tag{3.25}$$

Then (α',β') is a Lehmer pair with parameter $(D_1{X_1'}^2,-D_2^2{Y_1'}^2)$. Further, let $L_k(\alpha',\beta')(k\in\mathbb{N})$ denote the corresponding Lehmer numbers. Form (3.24) and (3.25), we have

$$D_2^{(m-2)/2}=Y_1'|L_{t'}(\alpha',\beta')|. \tag{3.26}$$

Since $(\alpha'^2-\beta'^2)^2=-D_1D_2^2X_1'^2Y_1'^2$, we see from (3.26) that the Lehmer number $L_{t'}(\alpha',\beta')$ has no primitive divisor. Therefore, using the same method as in the proof of Lemma 3.1, we can obtain the following lemma.

Lemma 3.3. If $D_1>1$, then all the solutions (x,m,n) of (1.2) with $2|m$ are given as follows:

(i) $t'=3,(D_1,D_2)=(7,3),(x,m,n)=(5,4,6)$.

(ii) $t'=3,(D_1,D_2)=(7,5),(x,m,n)=(17,2,9)$.

(iii) $t'=3,(D_1,D_2)=(15,7),(x,m,n)=(33,2,12)$.

(iv) $t'=1,Y_1'={D_2}^{(m-2)/2},(x,m,n)=(X_1',m,Z_1')$.

Let $N_2(D_1,D_2)$ denote the number of solutions (x,m,n) of (1.2) with $2|m$. By Lemma 3.3, we have:

Lemma 3.4. If $D_1>1$, then $N_2(D_1,D_2)\le 1$ except for the following cases:

(i) $N_2(7,3)=2,(x,m,n)=(1,2,2)$ and $(5,4,6)$.

(ii) $N_2(7,5)=2,(x,m,n)=(1,2,3)$ and $(17,2,9)$.

(iii) $N_2(15,7)=2,(x,m,n)=(1,2,4)$ and $(33,2,12)$.

§4. Proof of Theorem B

Lemma 4.1.([12]) The equation

$$\frac{x^n+1}{x+1}=y^2,x,y,z\in\mathbb{N},x>1,n>1$$

has no solution (x,y,n).

Lemma 4.2. The equation

$$2^{2r-3}-2^r+1=97^s,r,s\in\mathbb{N},r\ge 5 \tag{4.1}$$

has only the solution $(r,s)=(5,1)$.

Proof. By (4.1), we have

$$2(2^{r-2}-1)^2=97^s+1. \tag{4.2}$$

Since $r\ge 5$ and $2^{r-2}-1$ has an odd prime divisor p with $p\equiv 3(\text{mod } 4)$, if $2|s$, then (4.2) is impossible. So we have $2\nmid s$.

Since $2\nmid s$ and $97+1=2.7^2$, we see from (4.2) that $7|2^{r-2}-1$ and

$$\frac{97^s+1}{97+1}=\left(\frac{2^{r-2}-1}{7}\right)^2. \tag{4.3}$$

Apply Lemma 4.1 to (4.3), we get $s=1$ and $r=5$. Thus, the lemma is proved.

Lemma 4.3. The equation

$$7x^2+25^{2y}=2^{z+2},x,y,z\in\mathbb{N} \tag{4.4}$$

has no solution (x,y,z).

Proof. We suppose that (4.4) has a solution (x,y,z). Then the equation

$$7X^2+25Y^2=2^{Z+2},X,Y,Z\in\mathbb{Z},\gcd(X,Y)=1,Z>0 \tag{4.5}$$

has the solution

$$(X, Y, Z) = (x, 5^{2y-1}, z). \tag{4.6}$$

Since $(X_1, Y_1, Z_1) = (1, 1, 3)$ is the least solution of (4.5), apply Lemma 2.13 to (4.6), we have

$$z = 3t, t \in \mathbb{N}, 2 \nmid t, t > 1, \tag{4.7}$$

$$\frac{x\sqrt{7} + 5^{2k-1}\sqrt{-25}}{2} = \lambda_1 \left(\frac{\sqrt{7} + \lambda_2\sqrt{-25}}{2} \right)^t, \lambda_1, \lambda_2 \in \{\pm 1\}. \tag{4.8}$$

Let

$$\alpha = \frac{\sqrt{7} + \sqrt{-25}}{2}, \quad \beta = \frac{\sqrt{7} - \sqrt{-25}}{2}. \tag{4.9}$$

Then (α, β) is a Lehmer pair with parameter $(7, -25)$. Further, let $L_k(\alpha, \beta)(k \in \mathbb{N})$ denote the corresponding Lehmer numbers. By (4.8) and (4.9), we have

$$5^{2k-1} = |L_t(\alpha, \beta)|. \tag{4.10}$$

Since $(\alpha^2 - \beta^2) = -7 \cdot 25$, we see from (4.10) that the Lehmer number $L_t(\alpha, \beta)$ has no primitive divisor. Therefore, by Lemmas 2.18 and 2.19, we get $t \leq 5$. However, since $L_5(\alpha, \beta) = -55$ and $L_3(\alpha, \beta) = -1$, (4.10) is impossible. Thus, the lemma is proved.

Using the same method as in the proof of Lemma 4.3, we can obtain the following lemma.

Lemma 4.4. The equation

$$15x^2 + 49^{2y} = 2^{z+2}, \quad x, y, z \in \mathbb{N}$$

has no solution (x, m, n).

Lemma 4.5. If D_1 and D_2 satisfy (3.7), then $N_2(D_1, D_2) = 0$.

Proof. Under the assumption, we suppose that (1.2) has a solution (x, m, n) with $2|m$. Since $2 \nmid D_1D_2x$ and $D_2^m \equiv 1 (\mathrm{mod}\ 8)$, we have $D_1 \equiv D_1x^2 \equiv 2^{n+2} - D_2^m \equiv 7 (\mathrm{mod}\ 8)$. Hence, by (3.7), we get $\lambda = 1$ and

$$D_1X_1^2 = 2^{Z_1} - 1, \ D_2 = 3.2^{Z_1} + 1, \ Z_1 \geq 3. \tag{4.11}$$

By Lemmas 4.3 and 4.4, the lemma is true for $Z_1 \in \{3, 4\}$, We may therefore assume that $Z_1 \geq 5$.

Since $2|m$, we see from (1.2) that equation

$$X'^2 + D_1Y'^2 = 2^{Z'+2}, \ X', Y', Z' \in \mathbb{Z}, \ \gcd(X', Y') = 1, \ Z' > 0 \tag{4.12}$$

has the solution

$$(X', Y', Z') = (D_2^{m/2}, x, n). \tag{4.13}$$

Apply Lemma 2.13 to (4.13), we have

$$n = Z_1't, t \in \mathbb{N}, \tag{4.14}$$

$$\frac{D_2^{m/2} + x\sqrt{-D_1}}{2} = \lambda_1 \left(\frac{X_1' + \lambda_2 Y_1'\sqrt{-D_1}}{2} \right)^t, \lambda_1, \lambda_2 \in \{\pm 1\}, \tag{4.15}$$

where (X_1', Y_1', Z_1') is the least solution of (4.12).

Since $1^2 + D_1 X_1^2 = 2^{Z_1}$, by Lemma 2.17, the least solution (X_1', Y_1', Z_1') of (4.12) satisfies either

$$(X_1', Y_1', Z_1') = (1, X_1, Z_1 - 2) \tag{4.16}$$

or

$${X_1'}^2 - D_1 {Y_1'}^2 = 2\lambda, X_1' Y_1' = X_1, Z_1' = \frac{1}{2}(Z_1 - 2), \lambda \in \{\pm 1\}. \tag{4.17}$$

We first consider the case (4.16), then (4.14) and (4.15) can be written as

$$n = (Z_1 - 2)t, t \in \mathbb{N}, \tag{4.18}$$

$$\frac{D_2^{m/2} + x\sqrt{-D_1}}{2} = \lambda_1 \left(\frac{1 + \lambda_2 X_1 \sqrt{-D_1}}{2} \right)^t, \lambda_1, \lambda_2 \in \{\pm 1\}. \tag{4.19}$$

Form (4.19), we get $X_1 | x$, and hence, we have $x = X_1 y$, where $y \in \mathbb{N}$. Substituting it into (1.2), by (4.11) and (4.18), we get

$$D_1 x^2 + D_2^m = (2^{Z_1} - 1)y^2 + D_2^m = 2^{n+2} = 2^{(Z_1 - 2)t + 2}. \tag{4.20}$$

Let

$$\alpha = \frac{1 + X_1\sqrt{-D_1}}{2}, \quad \beta = \frac{1 - X_1\sqrt{-D_1}}{2}. \tag{4.21}$$

By (4.11), (4.19) and (4.21), we have

$$D_2^{m/2} = (3.2^{Z_1} + 1)^{m/2} = \lambda_1(\alpha^t + \beta^t). \tag{4.22}$$

Since $\alpha + \beta = 1$ and $\alpha\beta = 2^{Z_1 - 2}$, by Lemma 2.1, we have

$$\alpha^t + \beta^t = \sum_{i=0}^{[t/2]} (-1)^i \left[{t \atop i} \right] 2^{(Z_1 - 2)i}. \tag{4.23}$$

Since $\binom{t}{0} = 1$, $\binom{t}{1} = t$ and $Z_1 \geq 5$, compare (4.22) and (4.23) we obtain $\lambda_1 = 1$ and

$$t \equiv \begin{cases} 4 \pmod 8), \text{ if } 2 \| m; \\ 0 \pmod 8), \text{ if } 4 | m. \end{cases} \tag{4.24}$$

Since $4 | t$ by (4.24), we have $t = 4s$, where $s \in \mathbb{N}$. Hence, by (4.20), we get

$$2^{2(Z_1 - 2)s + 1} + D_2^{m/2} = Af^2, 2^{2(Z_1 - 2)s + 1} - D_2^{m/2} = Bg^2, \tag{4.25}$$

$$AB = 2^{Z_1} - 1 = D_1 X_1^2, A, B, f, g \in \mathbb{N}, \gcd(A, B) = \gcd(f, g) = 1.$$

From (4.25), we have

$$Af^2 + Bg^2 = 2^{2(Z_1 - 2)s + 2}. \tag{4.26}$$

By Lemma 2.13, we see from (4.16) that the equation

$${X''}^2 + (2^{Z_1} - 1){Y''}^2 = 2^{Z'' + 2}, X'', Y'', Z'' \in \mathbb{Z}, \gcd(X'', Y'') = 1, Z'' > 0 \tag{4.27}$$

is bound to have a solution (X'', Y'', Z'') with $Z'' = 2(Z_1 - 2)s$.
Therefore, since $AB = 2^{Z_1} - 1$, by Lemma 2.16, we get from (4.26) that

$$(A, B) = (1, 2^{Z_1} - 1) \ \text{ or } \ (2^{Z_1} - 1, 1). \tag{4.28}$$

On the other hand, by (4.25), we have

$$Af^2 - Bg^2 = 2D_2^{m/2} = 2(3 \cdot 2^{Z_1} + 1)^{m/2}. \tag{4.29}$$

Since $2^{Z_1} \equiv (\bmod AB)$ and $D_2 \equiv 3 \cdot 2^{Z_1} + 1 \equiv 4(\bmod 2^{Z_1} - 1)$, we have $(D_2/A) = (D_2/B) = 1$, where $(*/*)$ is the Jacobi symbol. Hence, by (4.25), we get

$$1 = \left(\frac{-2B}{A}\right) = \left(\frac{-2}{A}\right), \ 1 = \left(\frac{2A}{B}\right) = \left(\frac{2}{B}\right). \tag{4.30}$$

The combination of (4.28) and (4.30) yields $(A, B) = (1, 2^{Z_1} - 1)$. Substituting it into the first equality of (4.25), we have

$$f^2 - D_2^{m/2} = 2^{2(Z_1-2)s+1}. \tag{4.31}$$

If $4|m$, then from (4.31) we get $f + D_2^{m/4} = 2^{2(Z_1-2)s}$ and $f - D_2^{m/4} = 2$, whence we obtain

$$D_2^{m/4} = 2^{2(Z_1-2)s-1} - 1. \tag{4.32}$$

Since $2(Z_1 - 2)s - 1 \geq 5$, apply Lemma 2.8 to (4.32), we get $m = 4$ and $D_2 = 3 \cdot 2^{Z_1} + 1 = 2^{2(Z_1-2)s-1} - 1$. But, since $Z_1 \geq 5$, it is impossible. So we have $2 \parallel m$.

Since $2 \parallel m$, we see from (4.24) that $4 \parallel t$ and $t = 4s$, where $s \in \mathbb{N}$ with $2 \nmid s$. Since $\alpha^4 + \beta^4 = 1 - 2^{Z_1} + 2^{2Z_1-3}$ by (4.21), apply Lemma 2.1 to (4.22), we have $\lambda_1 = 1$ and

$$\begin{aligned} D_2^{m/2} &= (3 \cdot 2^{Z_1} + 1)^{m/2} = \alpha^t + \beta^t = (\alpha^4 + \beta^4)\left(\frac{(\alpha^4)^s + (\beta^4)^s}{\alpha^4 + \beta^4}\right) \\ &= (1 - 2^{Z_1} + 2^{2Z_1-3}) \sum_{j=0}^{(s-1)/2} (-1)^j \left[\begin{smallmatrix} s \\ j \end{smallmatrix}\right] (1 - 2^{Z_1} + 2^{2Z_1-3})^{s-2j-1} 2^{4(Z_1-2)j}. \end{aligned} \tag{4.33}$$

By (4.33), we get $2^{2Z_1-3} - 2^{Z_1} + 1 | (3 \cdot 2^{Z_1} + 1)^{m/2}$. Let $d = \gcd(2^{2Z_1-3} - 2^{Z_1} + 1, 3 \cdot 2^{Z_1} + 1)$. Then we have $d|97$. Hence, we get

$$2^{2Z_1-3} - 2^{Z_1} + 1 = 97^k, k \in \mathbb{N}. \tag{4.34}$$

Apply Lemma 4.2 to (4.34), we obtain $Z_1 = 5$ and $k = 1$. It implies that $(D_1, D_2) = (31, 97)$. Since (1.2) has a solution $(x, m, n) = (15, 2, 12)$ for $(D_1, D_2) = (31, 97)$, by Lemma 3.4, we get $N_2(31, 97) = 1$. Moreover, if $Z_1 > 5$, then (1.2) has no solution (x, m, n) with $2|m$ for the case (4.16).

We next consider the case (4.17). If $2 \nmid t$ for (4.14), then form (4.15) we get $X_1' | D_2^{m/2} = (3 \cdot 2^{Z_1} + 1)^{m/2}$. Since $X_1' | X_1$, $X_1 | 2^{Z_1} - 1$ and $\gcd(2^{Z_1} - 1, 3 \cdot 2^{Z_1} + 1) = 1$, we get $X_1' = 1$. Substituting it into (4.17), we obtain $Y_1' = 1$ and $D_1 = 3 \neq 2^{Z_1} - 1$, a contradiction. So we have $2|t$. Let $t = 2t'$, where $t' \in \mathbb{N}$. Since

$$\left(\frac{X_1' + \lambda_2 Y_1' \sqrt{-D_1}}{2}\right)^2 = \lambda \frac{1 + \lambda\lambda_2 X_1 \sqrt{-D_1}}{2}$$

by (4.17), we get from (4.14) and (4.15) that

$$n=(Z_1-2)t', t'\in\mathbb{N}, \tag{4.35}$$

$$\frac{D_2^{m/2}+x\sqrt{-D_1}}{2}=\lambda_1\lambda^{t'}\left(\frac{1+\lambda\lambda_2X_1\sqrt{-D_1}}{2}\right)^{t'}, \lambda,\lambda_1,\lambda_2\in\{\pm1\}. \tag{4.36}$$

Obviously, (4.35) and (4.36) are equal to (4.18) and (4.19) respectively. Thus, by the conclusion of case (4.16), the lemma is proved.

Proof of Theorem B. Let (x,m,n) be a solution of (1.2). Then we have $D_1\equiv 7 \pmod 8$ if $2\nmid m$. Therefore, by Lemmas 3.2 and 4.5, if $N_1(D_1,D_2)>0$, then $N_1(D_1,D_2)\leqslant 2$ except for $N(3,5)=N(5,3)=4$ and $N(13,3)=N(31,97)=3$. Further, by Lemmas 3.1, 3.4 and 4.5, if $N_2(D_1,D_2)>0$, then $N(D_1,D_2)\leqslant 2$. Thus, the theorem is proved.

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